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LETTER TO THE EDITOR

On tunnelling in the cubic potential

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Abstract. The tunnelling rate in real time in the semiclassical limit has been calculated for arbitrary energy levels in the cubic potential. For the ground state it agrees well with the result found by the instanton method.

In many areas of solid state physics there is a need to calculate the probability of quantum mechanical tunnelling of an object trapped in a metastable potential well [1-3]. The cubic potential is of particular importance. It is given by the following formula

$$V(x)=\tfrac{1}{2}m\omega^2x^2-\tfrac{1}{6}m\lambda x^3 \tag{1}$$

where x is the coordinate of the particle of mass m under consideration and ω is the frequency of small oscillations in the metastable well. There had been a great deal of effort put into calculating the lifetime in the above potential well. Most calculations deal with the ground state. The theory developed by Lapedes and Mottola [4] can allow for calculations of the lifetime for all states, not only the ground state. Their theory has been developed in imaginary time. In the present letter we present calculations for the lifetime of an arbitrary state in the potential well given by (1), working in real time. This was possible due to some properties of elliptic functions.

For our purpose let us use the Feynman propagator which describes the evolution of a particle from the initial state x_i to the final state x_f defined as

$$K(x_i, x_f, T)=\int_{x(0)=x_i}^{x(T)=x_f} \mathrm{D}x(t)\exp\left(\frac{\mathrm{i}}{\hbar}S[x]\right) \tag{2}$$

where $S[x]=\int_0^T \mathrm{d}t\, L(x,\dot{x})$ is the action of our system and

$$L=L(x,\dot{x})=\tfrac{1}{2}m\dot{x}^2-V(x). \tag{3}$$

In the semiclassical limit the above functional is dominated by the stationary points $x_c(t)$ obeying the equations

$$m\ddot{x}_c+\frac{\mathrm{d}}{\mathrm{d}x}V(x_c)=0 \tag{4}$$

along with boundary conditions $x_c(0)=x_c(T)=x_c$.

After integrating (4) for the potential (1), one finds

$$x_c(t)=(c-b)cn^2\sqrt{\frac{\lambda}{3g^2}}(t-t_0)+b \tag{5}$$

with the property $x_c(t+T)=x_c(t)$. Here $g=2/\sqrt{a-c}$, and

$$T=\sqrt{\frac{3g^2}{\lambda}}(2\mu K+2\mathrm{i}\nu K') \tag{6}$$

where μ, $\nu \in N$ and K and K' are complete elliptic integrals of the first kind. The values of a, b and c are given by

$$\begin{aligned} a&=\frac{\omega^2}{\lambda}(1+2\cos\tfrac{1}{3}\phi)\\ b&=\frac{\omega^2}{\lambda}(1-\cos\tfrac{1}{3}\phi+\sqrt{3}\sin\tfrac{1}{3}\phi)\\ c&=\frac{\omega^2}{\lambda}(1-\cos\tfrac{1}{3}\phi-\sqrt{3}\sin\tfrac{1}{3}\phi) \end{aligned} \tag{7}$$

where $\phi=\cos^{-1}(1-2\varepsilon/V_0)$, V_0 is the potential barrier height and ε is the energy of the tunnelling particle. With the above solution we are able to calculate the action $S[x_c]$. Because of the imaginary period of the elliptic functions, it will consist of two parts: real and imaginary. We have

$$S[x_c]=\mu S_1+\mathrm{i}\nu S_2 \tag{8}$$

where

$$S_1=-\frac{4m\lambda}{45}\left(\frac{3\omega^2}{\lambda}\right)^2\sqrt{\frac{3g^2}{\lambda}}\left[\left(a-\frac{1}{3}\frac{\omega^2}{\lambda}\frac{\varepsilon}{V_0}\right)K+(c-a)E\right] \tag{9}$$

$$S_2=-\frac{4m\lambda}{45}\left(\frac{3\omega^2}{\lambda}\right)^2\sqrt{\frac{3g^2}{\lambda}}\left[\left(a-\frac{1}{3}\frac{\omega^2}{\lambda}\frac{\varepsilon}{V_0}\right)K'+(c-a)(K'-E')\right]. \tag{10}$$

Here E and E' are the complete elliptic integrals of the second kind. The rest of our calculation is standard and we refer to Rajaraman [5]. We only quote the result here:

$$\begin{aligned} G(\varepsilon)&\equiv\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\exp\left(\frac{\mathrm{i}}{\hbar}\varepsilon T\right)\int \mathrm{d}x_0\,K(x_0,x_0,T)\\ &=\frac{1}{2\hbar\sqrt{m}}\sum_{\text{paths}}\exp(\mathrm{i}\theta_{\text{paths}})T_{\text{path}}\exp\left(\frac{\mathrm{i}}{\hbar}W(T_{\text{path}})\right). \end{aligned} \tag{11}$$

The summation in (11) over paths was done along the general lines described by Lapedes and Mottola [4], with the result

$$G(\varepsilon)=\frac{-1}{2\hbar\sqrt{m}}\frac{T_1\exp[(\mathrm{i}/\hbar)W_1]+\mathrm{i}T_2\exp(W_2/\hbar)}{1+\exp[(\mathrm{i}/\hbar)W_1]+\exp(W_2/\hbar)} \tag{12}$$

where

$$T_1=2\sqrt{\frac{3g^2}{\lambda}}K \qquad T_2=2\sqrt{\frac{3g^2}{\lambda}}K'$$

and

$$W_1=2\sqrt{m}\int_c^b\sqrt{2\varepsilon-2V(x)}\,\mathrm{d}x \qquad W_2=2\sqrt{m}\int_b^a\sqrt{2V(x)-2\varepsilon}\,\mathrm{d}x<0.$$

The decay rate to first order in the exponentially small $\exp(W_2/\hbar)$ is given by

$$\Gamma_n = \frac{\omega_n(\varepsilon_n^0)}{\pi}\exp\left(\frac{1}{\hbar}W_2(\varepsilon_n^0)\right) \tag{13}$$

where

$$\omega_n(\varepsilon_n^0) = \frac{2\pi}{T_1(\varepsilon_n^0)}.$$

To compare our results with existing results we have calculated Γ_n for the ground state. We have found

$$\Gamma_0 = \frac{36\omega}{\pi}\exp(47/120)\sqrt{\frac{2V_0}{3\hbar\omega}}\exp-(36V_0/5\hbar\omega). \tag{14}$$

This is in agreement with Caldeira instanton result [6] except for the prefactor which in our calculations is 3.34 times larger.

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